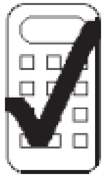
SJBC Further Maths Transition work

Transitioning to A-Level Further Mathematics requires a strong foundation in mathematics, including proficiency in A-Level Mathematics. Further Mathematics extends upon the concepts and topics covered in A-Level Mathematics, introducing more advanced topics and techniques. Here are some steps you can take to transition to A-Level Further Mathematics:

1. Solidify your A-Level Mathematics knowledge: Ensure that you have a firm grasp of the topics covered in A-Level Mathematics, including algebra, calculus, trigonometry, and coordinate geometry. Review and practice these concepts to strengthen your understanding.
2. Understand the A-Level Further Mathematics syllabus: Familiarize yourself with the syllabus for A-Level Further Mathematics, offered by your exam board (e.g., Edexcel, AQA, OCR). Identify the specific topics you will be studying, such as complex numbers, matrices, differential equations, and further calculus.
3. Obtain suitable textbooks and resources: Invest in textbooks specifically designed for A-Level Further Mathematics. These resources often provide comprehensive coverage of the syllabus, examples, and practice problems. Additionally, look for online resources, video lectures, and websites that can supplement your learning.
4. Seek guidance and additional support: Given the advanced nature of Further Mathematics, it's beneficial to seek guidance from your mathematics teacher or consider hiring a tutor with expertise in Further Mathematics. They can provide clarification, guidance, and additional resources to support your learning.
5. Engage with challenging problems: Further Mathematics requires advanced problem-solving skills. Engage with challenging problems from textbooks, past papers, and online resources. Focus on solving problems that involve multiple mathematical concepts, requiring creativity and critical thinking.
6. Embrace independent learning: Further Mathematics often requires self-study and independent learning. Develop good study habits, manage your time effectively, and create a study plan that includes regular practice, revision, and exposure to new concepts.

8. Stay motivated and persevere: A-Level Further Mathematics can be challenging, but maintaining motivation and perseverance are crucial. Embrace the challenges, view them as opportunities for growth, and seek support when needed. Consistent effort and a positive mindset will help you succeed.

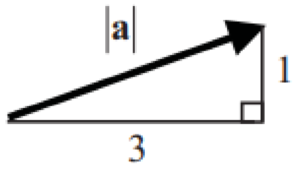
Transitioning to A-Level Further Mathematics requires dedication, a strong mathematical foundation, and a passion for the subject. By following these steps and committing to your studies, you'll be well-prepared to tackle the advanced concepts and techniques of A-Level Further Mathematics.



Magnitude of a Vector

The magnitude, or modulus, of a vector is just a term for its length. The length of a column vector can be found using Pythagoras' theorem. For example, consider the vector $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

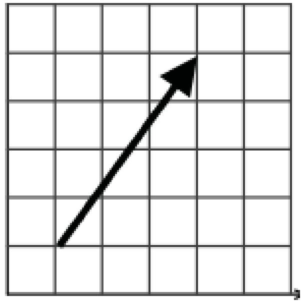
The notation for the length of **a** is $|\mathbf{a}|$



Using Pythagoras' theorem $|\mathbf{a}|^2 = 3^2 + 1^2 \Rightarrow \; = \sqrt{10}$.

A. Find the magnitudes of the following vectors, leaving your answer in surd form if appropriate.

1.



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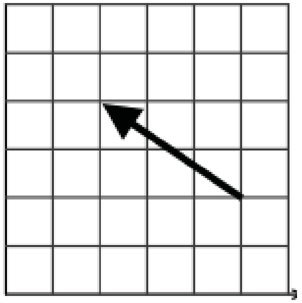

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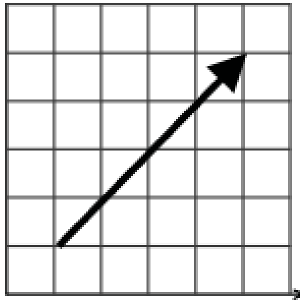

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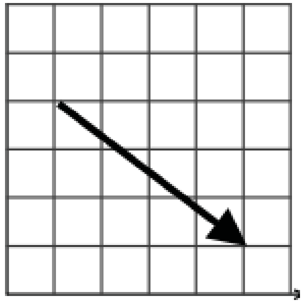


It is useful if you can calculate the length of a vector without drawing a diagram.
Pythagoras' theorem gives the following formula.

$$\left|\begin{pmatrix} a \\ b \end{pmatrix}\right| = \sqrt{a^2+b^2}.$$

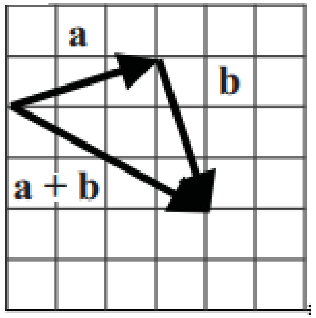
E.g. $\left|\begin{pmatrix} 4 \\ -2 \end{pmatrix}\right| = \sqrt{4^2+(-2)^2} = \sqrt{20} \quad = 2\sqrt{5}$

B. Find the lengths of the following vectors, leaving your answer in surd form if appropriate.

1. $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **2.** $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ **3.** $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$ **4.** $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **5.** $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ **6.** $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **7.** $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

8. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **9.** $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **10.** $\begin{pmatrix} 7 \\ 24 \end{pmatrix}$ **11.** $\begin{pmatrix} -7 \\ -1 \end{pmatrix}$ **12.** $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ **13.** $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$ **14.** $\begin{pmatrix} 8 \\ -4 \end{pmatrix}$

C. The diagram shows three vectors, **a**, **b** and **a + b**. Find the length of each vector and use Pythagoras's theorem to show that **a** is perpendicular to **b**.



Solution:

$\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\mathbf{a}+\mathbf{b} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

$|\mathbf{a}| = \sqrt{10}$, $\quad |\mathbf{b}| = \sqrt{10}$, $\quad |\mathbf{a}+\mathbf{b}| = \sqrt{20}$.

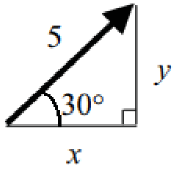
Using Pythagoras' theorem $\left(\sqrt{10}\right)^2 + \left(\sqrt{10}\right)^2 = 10+10 = 20 = \left(\sqrt{20}\right)^2$

therefore **a** and **b** are perpendicular.

Components

Sometimes we know the length and direction of a vector but need to know its column vector (component) form.

The vector of length 5 at 30° to the x-axis can be split into two components using trigonometry.



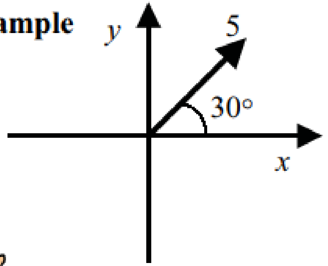
$\cos 30° = \frac{x}{5} \Rightarrow x = 5\cos 30 = 4.3$

$\sin 30° = \frac{y}{5} \Rightarrow y = 5\sin 30 = 2.5$

So the vector is $\begin{pmatrix} 4.3 \\ 2.5 \end{pmatrix}$

Find the following vectors as column vectors.
Give answers to one decimal place.

Example



Diagrams not to scale

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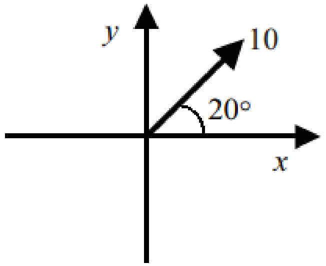

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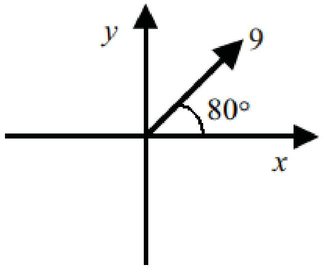

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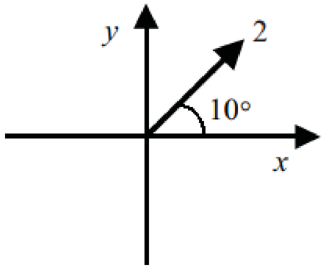

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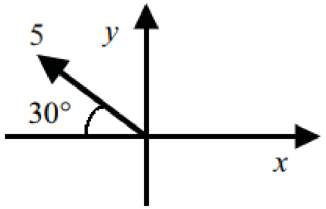

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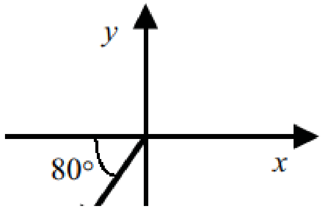

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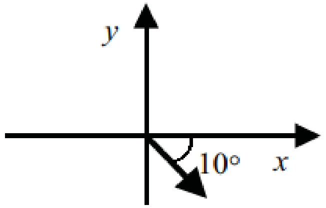

In the last six questions find each vector in column form and hence add the vectors together.

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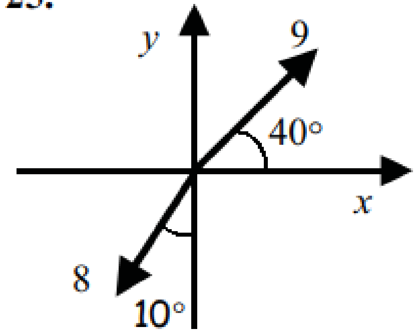

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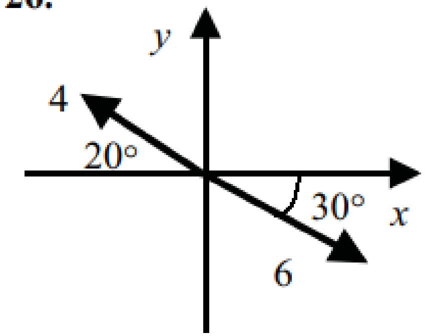

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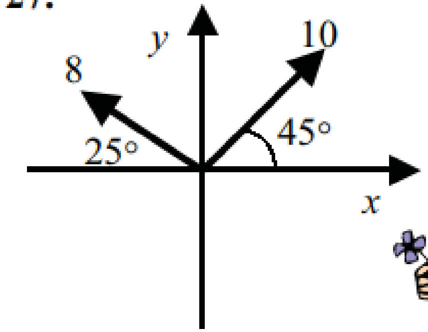

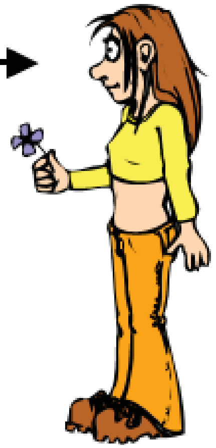

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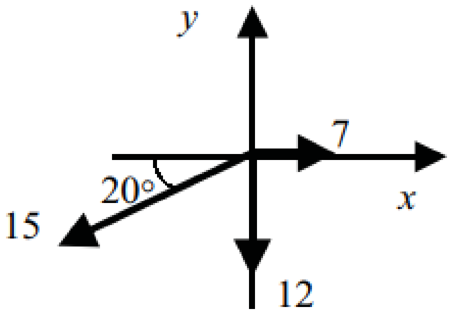

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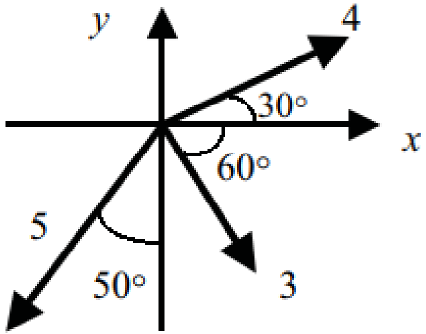

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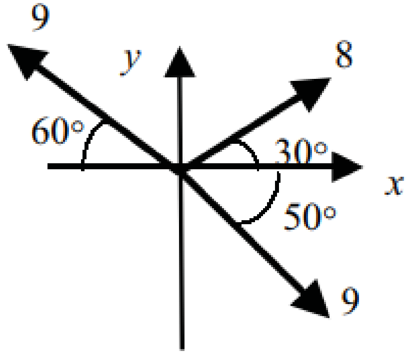


Page 8. Magnitude of a Vector

A. **1.** 5 **2.** $\sqrt{13}$ **3.** $4\sqrt{2}$ **4.** 5 **5.** 5 **6.** 4 **7.** $3\sqrt{5}$

8. $\sqrt{41}$ **9.** 5 **10.** $\sqrt{17}$ **11.** $\sqrt{2}$ **12.** $2\sqrt{2}$ **13.** $3\sqrt{2}$ **14.** $4\sqrt{2}$

15. $3\sqrt{5}$ **16.** $\sqrt{34}$

Page 9.

B. **1.** 5 **2.** 5 **3.** 13 **4.** $\sqrt{13}$ **5.** $2\sqrt{5}$ **6.** $\sqrt{2}$ **7.** $\sqrt{2}$

8. $\sqrt{5}$ **9.** $\sqrt{5}$ **10.** 25 **11.** $5\sqrt{2}$ **12.** $\sqrt{5}$ **13.** $5\sqrt{2}$ **14.** $4\sqrt{5}$

C. **1.** **(i)** $\sqrt{2},\sqrt{2},2$ **(ii)** $\sqrt{5},\sqrt{5},\sqrt{10}$ **(iii)** $\sqrt{13},\sqrt{13},\sqrt{26}$ **(iv)** $\sqrt{20},\sqrt{5},5$

(v) $\sqrt{17},\sqrt{17},\sqrt{34}$ **(vi)** $3\sqrt{5},\sqrt{5},5\sqrt{2}$ **(vii)** $2\sqrt{2},2\sqrt{2},4$ **(viii)** $2\sqrt{17},2\sqrt{17},2\sqrt{34}$

(ix) $4\sqrt{5}, \sqrt{5}, \sqrt{85}$

3. (a) $\begin{pmatrix} a+c \\ b+d \end{pmatrix}$ **(b)** $\sqrt{a^2+b^2}$ **(c)** $\sqrt{c^2+d^2}$ **(d)** $\sqrt{(a+c)^2+(b+d)^2}$

Page 11. Components

1. $\begin{pmatrix} 9.4 \\ 3.4 \end{pmatrix}$ **2.** $\begin{pmatrix} 1.6 \\ 8.9 \end{pmatrix}$ **3.** $\begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}$ **4.** $\begin{pmatrix} -4.3 \\ 2.5 \end{pmatrix}$ **5.** $\begin{pmatrix} -1.6 \\ -8.9 \end{pmatrix}$ **6.** $\begin{pmatrix} 2.0 \\ -0.3 \end{pmatrix}$ **7.** $\begin{pmatrix} -2.5 \\ 4.3 \end{pmatrix}$ **8.** $\begin{pmatrix} -8.9 \\ -1.6 \end{pmatrix}$

9. $\begin{pmatrix} 0.3 \\ -2.0 \end{pmatrix}$ **10.** $\begin{pmatrix} 3.4 \\ 9.4 \end{pmatrix}$ **11.** $\begin{pmatrix} -3.9 \\ 4.6 \end{pmatrix}$ **12.** $\begin{pmatrix} -6.9 \\ -4 \end{pmatrix}$

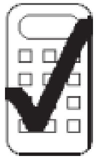
Page 12.

13. $\begin{pmatrix} 6.3 \\ 3.0 \end{pmatrix}$ **14.** $\begin{pmatrix} -1.7 \\ 2.5 \end{pmatrix}$ **15.** $\begin{pmatrix} -1.6 \\ -5.8 \end{pmatrix}$ **16.** $\begin{pmatrix} -6.0 \\ -0.5 \end{pmatrix}$ **17.** $\begin{pmatrix} 3.4 \\ 7.3 \end{pmatrix}$ **18.** $\begin{pmatrix} 9.8 \\ -6.9 \end{pmatrix}$ **19.** $\begin{pmatrix} 12.3 \\ -8.6 \end{pmatrix}$ **20.** $\begin{pmatrix} -1.6 \\ -8.9 \end{pmatrix}$

21. $\begin{pmatrix} 3.4 \\ 3.7 \end{pmatrix}$ **22.** $\begin{pmatrix} -3.2 \\ 6.2 \end{pmatrix}$ **23.** $\begin{pmatrix} -3.5 \\ -1.9 \end{pmatrix}$ **24.** $\begin{pmatrix} 3.7 \\ 4.7 \end{pmatrix}$ **25.** $\begin{pmatrix} 5.5 \\ -2.1 \end{pmatrix}$ **26.** $\begin{pmatrix} 1.4 \\ -1.6 \end{pmatrix}$ **27.** $\begin{pmatrix} -0.2 \\ 10.5 \end{pmatrix}$ **28.** $\begin{pmatrix} -7.1 \\ -17.1 \end{pmatrix}$

29. $\begin{pmatrix} 1.2 \\ -3.8 \end{pmatrix}$ **30.** $\begin{pmatrix} 8.2 \\ 4.9 \end{pmatrix}$

Gradient of Curves 2



1. The gradient function on a curve has a special notation : $\frac{dy}{dx}$.

It can be shown that for the curve $y = ax^n$, $= anx^{n-1}$.

Find $\frac{dy}{dx}$ for the following curves.

(a) $y = 2x^2$ **(b)** $y = 4x^3$ **(c)** $y = 3x^5$ **(d)** $y = 12x^{10}$ **(e)** $y = -4x^2$

(f) $y = -3x^{-1}$ **(g)** $y = 3x$ **(h)** $y = 4x$ **(i)** $y = -5x$ **(j)** $y = -4x^2$

(k) $y = -3x^{-2}$ **(l)** $y = 12x - 1$ **(m)** $y = 3x + 2$ **(n)** $y = x^2 + x$ **(o)** $y = 2x^2 + 4x^3$

(p) $y = 2x^2 + 3x + 4$ **(q)** $y = 5x^2 - 6x - 2$ **(r)** $y = 1 - x^2$

(s) $y = 2x - x^3 + 4$ **(t)** $y = (x-1)(x+1)$ **(u)** $y = x(x+3)(x-1)$

(v) $y = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$ **(w)** $y = x - 3x^{-2}$ **(x)** $y = \frac{3}{x^2} - \frac{2}{x} + 3$

(y) $y = 4x^2 + (2x-1)^2$ **(z)** $y = \left(x - \frac{1}{x}\right)^2$.

2. Find $\frac{dy}{dx}$ for the following, and by putting $\frac{dy}{dx} = 0$,
find the coordinates of the points on the curves where the gradient is zero.

(a) $y = x^2 - 2x + 4$ **(b)** $y = x^2 + 4x + 4$ **(c)** $y = 3x^2 - 6x$

(d) $y = 3 - x^2$ **(e)** $y = 2x - x^2 - 10$ **(f)** $y = x^3 - 3x$

(g) $y = 2x^3 - 3x^2 - 12x + 2$ **(h)** $y = 9x - 3x^2 - x^3$ **(i)** $y = x^3 - x$

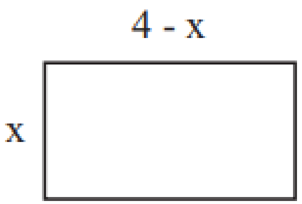
3. For each of the rectangles below

(a) find the area, A, in terms of x,

(b) find $\dfrac{dA}{dx}$,

(c) find the value of x which makes the area of the rectangle greatest.

(i)



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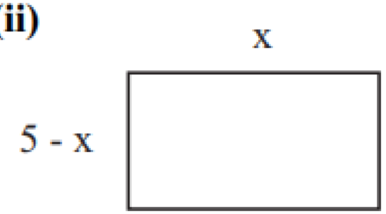

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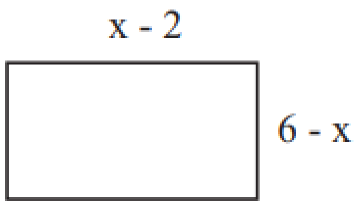
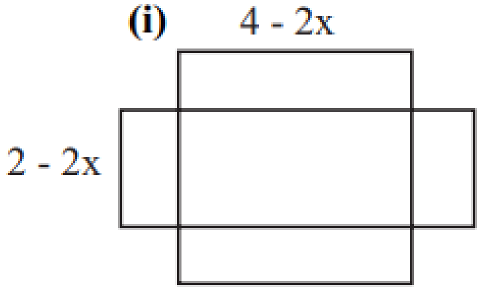


4. Each of the following diagrams shows a rectangle where a square (x by x) has been removed at the corners. The resulting shape can be folded to make an open box. For each one

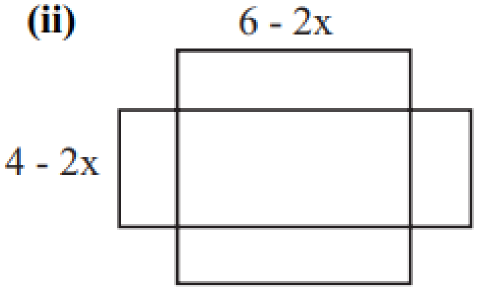
(a) find the volume of the box so formed, V, in terms of x,

(b) find $\dfrac{dV}{dx}$ and the value of x which makes the volume of the box greatest.

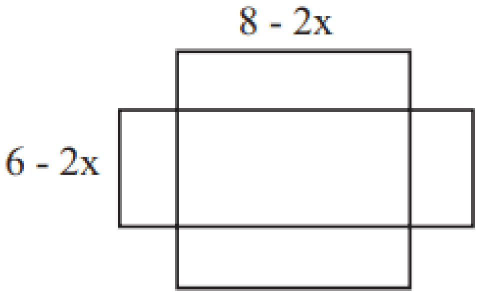
(i)

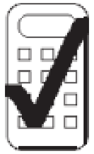


(ii)



(iii)





Integration 1

Indefinite Integrals

Integration is the opposite of differentiation. There is a simple rule for the integration of a polynomial:

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c \qquad n \neq -1$$

A. Use this rule to integrate the following,

1. $3x^2$	**2.** $4x^7$	**3.** $6x^{11}$	**4.** $4x^3$	**5.** $1.5x^{0.5}$
6. $x^2 + 4x + 3$	**7.** $3x^2 + 2x + 1$	**8.** $5x^2 - 3x - 2$	**9.** $7x^3 + x^4 - 3$	**10.** $16x^{-2} - x$
11. $3x^4 - x^{16}$	**12.** $2x^5 - x^{-6}$	**13.** $x^{10} + x^{20}$	**14.** $x^2 - 1$	**15.** $x^{-0.5}$
16. $x^3 + 4x^2 + 3x$	**17.** $x^4 + 2x^2$	**18.** $x(x - 1)$	**19.** $(x - 1)(x + 1)$	**20.** $(2x - 1)(3x + 1)$

Integration can be used to find the equation of a curve when the gradient function (i.e. $\frac{dy}{dx}$) is known.

Example

Suppose we know that, $\frac{dy}{dx} = 3x^2 + 1$, and that the curve passes through the point (1,5).

We can integrate the expression for $\frac{dy}{dx}$ to get y.

If $\frac{dy}{dx} = 3x^2 + 1$, then $y = \int 3x^2 + 1 \, dx \Rightarrow y = x^3 + x + c$.

We now substitute (1,5) into this expression to find c.

$5 = 1^3 + 1 + c$ therefore $c = 3$.

Notice the arbitrary constant, c, in this equation.
This must be included because when $x^3 + x + c$ is differentiated, it will always give $3x^2 + 1$ for all constants, c.

Integration 2

Definite Integrals

A definite integral has limits. The indefinite integral is found first of all, then the limits are substituted into the new expression, and finally the difference is found. The following example shows how.

$$\int_1^2 (2x^2+4x+5)\ \mathrm{d}x = \left[\tfrac{2}{3}x^3+2x^2+5x\right]_1^2$$

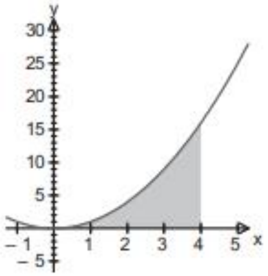
$$=\left(\tfrac{2}{3}\times 2^3+2\times 2^2+5\times 2\right)-\left(\tfrac{2}{3}\times 1^3+2\times 1^2+5\times 1\right)=15\tfrac{2}{3}$$

A. Find the following definite integrals.

1. $\int_1^2 x^2\ \mathrm{d}x$ **2.** $\int_2^3 3x^2+1\ \mathrm{d}x$ **3.** $\int_0^1 x^3\ \mathrm{d}x$ **4.** $\int_{-1}^2 5x\ \mathrm{d}x$ **5.** $\int_{-1}^1 x^2-1\ \mathrm{d}x$

6. $\int_0^2 2-x\ \mathrm{d}x$ **7.** $\int_0^1 x^2-x\ \mathrm{d}x$ **8.** $\int_{-2}^2 x\ \mathrm{d}x$ **9.** $\int_0^3 4x^3-2x\ \mathrm{d}x$ **10.** $\int_{-1}^0 x(x+1)\ \mathrm{d}x$

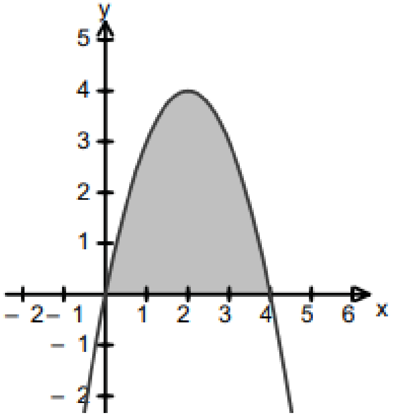
Definite integrals can be used to find areas between a curve and the x - axis. For example the shaded area in the diagram shown opposite, between the curve $y=x^2$ and the x - axis, between $x=1$ and $x=4$, is given by,



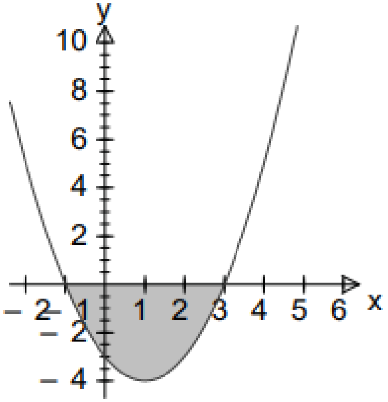
$$\int_1^4 x^2\ \mathrm{d}x = \left[\tfrac{1}{3}x^3\right]_1^4 = \tfrac{64}{3}-\tfrac{1}{3}=21$$

B. Find the areas shaded in the following diagrams

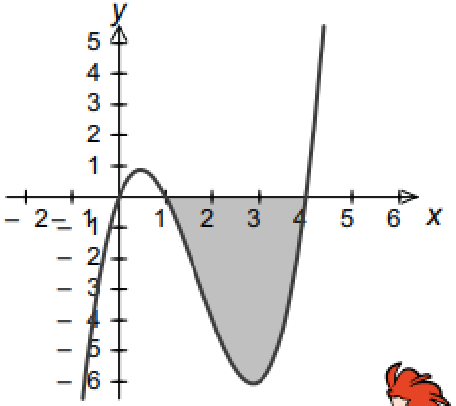
1. $y = 4x - x^2$



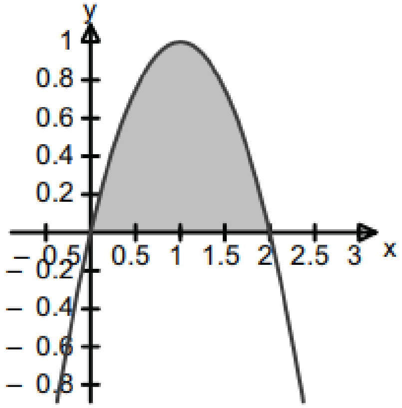
2. $y = x^2 - 2x - 3$



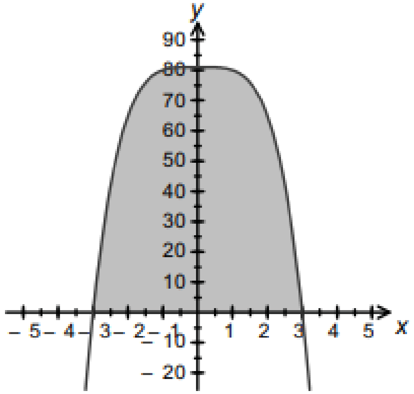
3. $y = x^3 - 5x^2 + 4x$



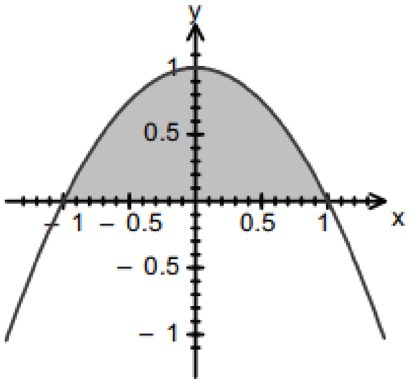
4. $y = x(2 - x)$



5. $y = 81 - x^4$



6. $y = 1 - x^2$



Page 41. Gradients of a Curve 2.

1. **(a)** $4x$ **(b)** $12x^2$ **(c)** $15x^4$ **(d)** $120x^9$ **(e)** $-8x$ **(f)** $3x^{-2}$ **(g)** 3

(h) 4 **(i)** -5 **(j)** $-8x$ **(k)** $6x^{-3}$ **(l)** 12 **(m)** 3 **(n)** $2x+1$

(o) $4x+12x^2$ **(p)** $4x+3$ **(q)** $10x-6$ **(r)** $-2x$ **(s)** $2-3x^2$

(t) $2x$ **(u)** $3x^2+4x-3$

(v) $-\frac{1}{x^2}-\frac{2}{x^3}-\frac{3}{x^4}$ **(w)** $1+\frac{6}{x^3}$ **(x)** $\frac{2}{x^2}-\frac{6}{x^3}$ **(y)** $16x-4$ **(z)** $2x + 2/x^3$

2. **(a)** $2x-2, x=1$, $y = 3$ **(b)** $2x+4, x=-2$, $y = 0$

(c) $6x-6, x=1$, $y = -3$ **(d)** $-2x, x=0$, $y = 3$

(e) $2-2x, x=1$, $y = -9$ **(f)** $3x^2-3, x=\pm 1$, $y = 0$ or -2

(g) $6x^2-6x-12, x=-1$ or 2, $y = 9$ or -18

Page 43. Integration 2.

A. **1.** $\frac{7}{3}$ **2.** 20 **3.** $\frac{1}{4}$ **4.** 7.5 **5.** $-{}^{4}/_{3}$ **6.** 2 **7.** $-\frac{1}{6}$

8. 0 **9.** 72 **10.** $-\frac{1}{6}$

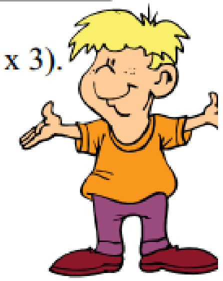
B. **1.** $10\frac{2}{3}$ **2.** $10\frac{2}{3}$ **3.** $11\frac{1}{4}$ **4.** $1\frac{1}{3}$ **5.** $388\frac{4}{5}$ **6.** $\frac{4}{3}$

Matrices 1.

Addition, Subtraction and Multiples.

> A matrix is a **rectangular** array of numbers.
> Each number in the matrix is known as an **element** of the matrix.
> The "size" of the matrix, determined by the number of its **rows** and **columns**, is called the **order** of the matrix.

A matrix that has 2 rows and 3 columns has an order "2 by 3" (written as 2 x 3).
Note that the **first** number in the order is always the number of **rows**.



	Matrix	Order	
Example 1	$\begin{pmatrix} 5 & 6 & 7 \\ 3 & 2 & 1 \end{pmatrix}$	2 x 3	
Example 2	$\begin{pmatrix} 6 \\ 4 \end{pmatrix}$	2 x 1	sometimes called a column matrix
Example 3	(-2, 3)	1 x 2	sometimes called a row matrix
Example 4	$\begin{pmatrix} 6 & 5 \\ 7 & 8 \end{pmatrix}$	2 x 2	sometimes called a square matrix of order 2

Question 1 Write down the order of each of the following matrices.

(a) $\begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{pmatrix}$ (d) $\begin{pmatrix} 12 \end{pmatrix}$ (e) $\begin{pmatrix} 7 & 5 & 3 & 1 \end{pmatrix}$

Matrix Addition is the process whereby two matrices **of the same order** are combined by **adding together corresponding elements**.

Example 5 Given $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

To find $\mathbf{A} + \mathbf{B}$, add the first elements $(1 + 5 = 6)$

then add the second elements $(2 + (-3) = -1)$

so $$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 1+5 \\ 2+(-3) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

Example 6 Given $\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -4 \\ 7 & 1 \end{pmatrix}$

then $$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 3+2 & 5+(-4) \\ 0+7 & (-1)+1 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 7 & 0 \end{pmatrix}$$

Question 2 Express each of the following as a single matrix.

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 2 & -1 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & -2 \end{pmatrix}$ (e) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ (f) $\begin{pmatrix} 6 & -3 \end{pmatrix} + \begin{pmatrix} -1 & 2 \end{pmatrix}$

(g) $\begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix}$ (h) $\begin{pmatrix} 3 & 2 \\ -1 & 0 \\ 7 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 5 & -1 \\ -4 & 2 \end{pmatrix}$ (i) $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ -3 & 6 \end{pmatrix}$

(j) $\begin{pmatrix} 7 & 5 & 3 \\ 2 & -4 & -6 \end{pmatrix} + \begin{pmatrix} 0 & 2 & -4 \\ 4 & -2 & 6 \end{pmatrix}$ (k) $\begin{pmatrix} 4 & 8 & 3 \\ -4 & -4 & -6 \end{pmatrix} + \begin{pmatrix} 1 & -2 & -4 \\ 4 & -2 & 6 \end{pmatrix}$

> The process of matrix addition can be extended to combining more than two matrices, provided **all the matrices involved have the same order**.

Example 7 Given $A = \begin{pmatrix} 3 & 5 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -4 \\ 7 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & 2 \\ 2 & 0 \end{pmatrix}$

then $\mathbf{A} + \mathbf{B} + \mathbf{C} = \begin{pmatrix} 3+2+(-1) & 5+(-4)+2 \\ 0+7+2 & (-1)+1+0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 9 & 0 \end{pmatrix}$

Question 3 Express each of the following as a single matrix.

(a) $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 5 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 5 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3 \end{pmatrix}$

> **Matrix Subtraction** is performed in the same manner as addition, except that the corresponding elements are subtracted. **The matrices must have the same order**.

Example 8 Given $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

To find **A - B**, subtract the first elements (1 - 5 = -4)

then subtract the second elements (2 - (-3) = 5)

so $$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 1-5 \\ 2-(-3) \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

Matrix Multiplication 1.

Example 1 The row matrix $\begin{pmatrix} 2 & 4 & 7 \end{pmatrix}$ and the column matrix $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

can be combined by **multiplying**

the **first** element in the row matrix by the **first** element in the column matrix (i.e. 2 x 3),
the **second** element in the row matrix by the **second** element in the column matrix (i.e. 4 x 5),
the **third** element in the row matrix by the **third** element in the column matrix (i.e. 7 x 1),
and **adding** the three **products** together, to give a 1 x 1 matrix.

Thus $$\begin{pmatrix} 2 & 4 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\times3+4\times5+7\times1 \end{pmatrix} = \begin{pmatrix} 33 \end{pmatrix}.$$

This method can be used to combine a row matrix and a column matrix, **with the same number of elements** in each, of any size.

Example 2 $$\begin{pmatrix} 5 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 5\times4+3\times(-2) \end{pmatrix} = \begin{pmatrix} 20-6 \end{pmatrix} = \begin{pmatrix} 14 \end{pmatrix}$$

Question 1 Use the method shown in Example 1 and Example 2 to combine the following row and column matrices to give a 1 x 1 matrix.

(a) $\begin{pmatrix} 7 & 2 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 & -4 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

(d) $\begin{pmatrix} 4 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (e) $\begin{pmatrix} 4 & -5 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (f) $\begin{pmatrix} -4 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(g) $\begin{pmatrix} 6 & 5 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \\ 0 \\ 4 \end{pmatrix}$ (h) $\begin{pmatrix} 6 & -5 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -7 \\ 0 \\ 4 \end{pmatrix}$ (i) $\begin{pmatrix} 6 & 5 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -7 \\ 0 \\ -4 \end{pmatrix}$

Question 2 Find the value of x in each of the following equations.

(a) $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ 4 \end{pmatrix} = (18)$ (b) $\begin{pmatrix} 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ -2 \end{pmatrix} = (34)$ (c) $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ x \end{pmatrix} = (29)$

(d) $\begin{pmatrix} 3 & 2x & 4 \end{pmatrix}\begin{pmatrix} x \\ 1 \\ 3 \end{pmatrix} = (52)$ (e) $\begin{pmatrix} 3x & 2 & -4 \end{pmatrix}\begin{pmatrix} 6 \\ x \\ 3 \end{pmatrix} = (48)$ (f) $\begin{pmatrix} 3 & 2x & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = (10x)$

(g) $\begin{pmatrix} 5 & 3 \end{pmatrix}\begin{pmatrix} -2x \\ x \end{pmatrix} = (28)$ (h) $\begin{pmatrix} 2 & x \end{pmatrix}\begin{pmatrix} 4 \\ x \end{pmatrix} = (33)$ (i) $\begin{pmatrix} 5 & x \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix} = (6)$

Complex numbers - Exercises with detailed solutions

1. Compute real and imaginary part of $z = \dfrac{i-4}{2i-3}$.
2. Compute the absolute value and the conjugate of
$$z = (1+i)^6, \qquad w = i^{17}.$$
3. Write in the "algebraic" form $(a+ib)$ the following complex numbers
$$z = i^5 + i + 1, \qquad w = (3+3i)^8.$$
4. Write in the "trigonometric" form $(\rho(\cos\theta + i\sin\theta))$ the following complex numbers
$$a)8 \qquad b)6i \qquad c)\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^7.$$
5. Simplify
   (a) $\dfrac{1+i}{1-i} - (1+2i)(2+2i) + \dfrac{3-i}{1+i}$;
   (b) $2i(i-1) + \left(\sqrt{3}+i\right)^3 + (1+i)\overline{(1+i)}$.
6. Compute the square roots of $z = -1-i$.
7. Compute the cube roots of $z = -8$.
8. Prove that there is no complex number such that $|z| - z = i$.
9. Find $z \in \mathbb{C}$ such that
$$a)\overline{z} = i(z-1) \qquad b)z^2 \cdot \overline{z} = z \qquad c)|z+3i| = 3|z|.$$
10. Find $z \in \mathbb{C}$ such that $z^2 \in \mathbb{R}$.
11. Find $z \in \mathbb{C}$ such that
    (a) $\mathrm{Re}\,(z(1+i)) + z\overline{z} = 0$;
    (b) $\mathrm{Re}\left(z^2\right) + i\,\mathrm{Im}\,(\overline{z}(1+2i)) = -3$;
    (c) $\mathrm{Im}\,((2-i)z) = 1$.

Solutions

1. $z = \dfrac{i-4}{2i-3} = \dfrac{i-4}{2i-3} \cdot \dfrac{2i+3}{2i+3} = \dfrac{-2+3i-8i-12}{-4-9} = \dfrac{14}{13} + i\dfrac{5}{13}$ hence $\mathrm{Re}\,(z) = \dfrac{14}{13}$ and $\mathrm{Im}\,(z) = \dfrac{5}{13}$.

2. $z = (1+i)^6 = \left(\sqrt{2}(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4})\right)^6 = 8\left(\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2})\right) = -8i$. Hence $|z| = 8$ and $\bar{z} = 8i$.

   $w = i^{17} = i \cdot i^{16} = i \cdot (i^4)^4 = i \cdot (1)^4 = i$. Hence $|w| = 1$ and $\bar{w} = -i$.

3. $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ e $i^5 = i$ then $z = i + i + 1 = 1 + 2i$.

   For w, we write $3+3i$ in the trigonometric form. We have $3+3i = 3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, hence

$$(3+3i)^8 = 3^8 \cdot 2^4 \left(\cos\left(8 \cdot \frac{\pi}{4}\right) + i\sin\left(8 \cdot \frac{\pi}{4}\right)\right) = 16 \cdot 3^8\left(\cos 2\pi + i\sin 2\pi\right) = 16 \cdot 3^8.$$

4. If $z = a + ib$, $a, b \in \mathbb{R}$, its trigonometric form is

$$z = \rho\left(\cos\theta + i\sin\theta\right), \qquad \text{where } \rho := \sqrt{a^2+b^2} \text{ and } \theta \text{ is such that } \cos\theta = \frac{a}{\rho},\ \sin\theta = \frac{b}{\rho}.$$

   a) $a = 8$, $b = 0$, $\cos\theta = 1$ e $\sin\theta = 0$. Hence $8 = 8\left(\cos 0 + i\sin 0\right)$.
   b) $6i = 6\left(0 + i\right) = 6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.
   c) We use the de Moivre's Formula:

$$\left(\cos(\frac{\pi}{3}) - i\sin(\frac{\pi}{3})\right)^7 = \cos\frac{7\pi}{3} - i\sin\frac{7\pi}{3} = \cos 2\pi + \frac{\pi}{3} - i\sin 2\pi + \frac{\pi}{3} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}.$$

5. (a) We compute

$$\frac{1+i}{1-i} - (1+2i)(2+2i) + \frac{3-i}{1+i} = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} - (1+2i)(2+2i) + \frac{3-i}{1+i} \cdot \frac{1-i}{1-i}$$

$$= i - 2 - 2i - 4i + 4 + \frac{3-1-3i-i}{2} = i + 2 - 6i + \frac{2-4i}{2} = 2 - 5i + 1 - 2i = 3 - 7i.$$

   (b) Since

$$\left(\overline{\sqrt{3}+i}\right)^3 = \left(\sqrt{3}-i\right)^3 = \left(\sqrt{3}-i\right)^2\left(\sqrt{3}-i\right) = \left(3 - 1 - 2i\sqrt{3}\right)\left(\sqrt{3}-i\right)$$

$$= \left(2 - 2i\sqrt{3}\right)\left(\sqrt{3}-i\right) = 2\sqrt{3} - 2i - 6i - 2\sqrt{3} = -8i,$$